The Inductance Matrix of a Multiconductor Transmission Line in Multiple Magnetic Media

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Abstract **—Consider a multiconductor transmission line consisting of N_c conducting cylinders in inhomogeneous media consisting of N_d homogeneous regions with permeabilities μ_i and permittivities ϵ_i. The inductance matrix $[L]$ for the line is obtained by solving the magnetostatic problem of N_c conductors in N_d regions with permeabilities μ_i. The capacitance matrix $[C]$ for the line is obtained by solving the electrostatic problem of N_c conductors in N_d regions with permittivities ϵ_i. It is shown that $[L]=\mu_0\epsilon_0[C']^{-1}$, where $[C']$ is the capacitance matrix of an auxiliary electrostatic problem of N_c conductors in N_d regions with relative permittivities set equal to the reciprocals of the relative permeabilities of the magnetostatic problem, i.e., $\epsilon_i'/\epsilon_0=\mu_0/\mu_i$.**

I. Introduction

Fig. 1 shows a multiconductor transmission line consisting of N_c conductors and N_d insulating materials above a perfectly conducting ground plane. The system is uniform in the z direction (direction of propagation), and the cross-sectional shapes of the conductors and insulators are arbitrary. The insulators have arbitrary permeabilities μ_i and permittivities ϵ_i. An upper ground plane could be present and treated as an additional conductor in a manner similar to that of [1].

To the quasi-static approximation, the multiconductor transmission line is characterized by a capacitance matrix $[C]$, obtained from an electrostatic analysis, and an inductance matrix $[L]$, obtained from a magnetostatic analysis. The formulation of the problem for $[C]$ and a numerical algorithm for its computation are given in [1]. In the appendix of [1] it is shown that if the insulating matter is nonmagnetic, the inductance matrix of the line is given by

$$[L]=\mu_0\epsilon_0[C_0]^{-1}. \tag{1}$$

Here $[C_0]$ is the capacitance matrix of the multiconductor transmission line if all dielectric constants are set equal to 1.

In this paper we shall show that when the insulators are magnetic, a modified relationship holds. In particular, it is

$$[L]=\mu_0\epsilon_0[C']^{-1} \tag{2}$$

where $[C']$ is the capacitance matrix of the line if all relative permittivities are set equal to the reciprocals of the relative permeabilities, i.e.,

$$\frac{\epsilon_i'}{\epsilon_0}=\frac{\mu_0}{\mu_i}, \qquad i=1,2,\cdots,N_d. \tag{3}$$

Note that the auxiliary problem for finding $[C']$ has relative permittivities less than unity if the corresponding relative permeabilities are greater than unity.

Relationship (2) is strictly true only if all conductors are perfect. It assumes that all current in the magnetostatic problem flows on the surfaces of conductors. For actual conductors,

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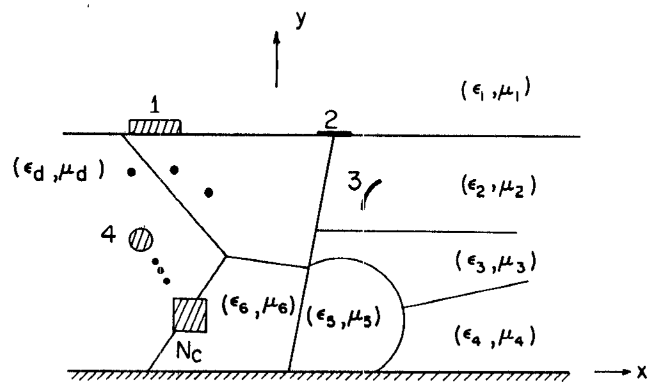

Fig. 1. A multiconductor transmission line in a multilayered dielectric and magnetic region above a ground plane.

depending on the frequency, some of the current flows internal to the conductors. Relationship (2) is then only approximate.

Although (2) can be inferred from the last two sentences in [2, sec. V], we have not seen an explicit proof in the literature. The purpose of this short paper is to give a simple proof of (2) for the multiconductor case with layered media.

II. Proof of (2)

The electrostatic problem from which $[C]$ is calculated is formulated in detail in [1]. We shall refer to that formulation when needed. The magnetostatic problem from which $[L]$ is calculated is an extension of the formulation for nonmagnetic media, given in the appendix of [1]. The formulation for magnetic media is given below.

The inductance matrix $[L]$ is an $N_c\times N_c$ matrix that satisfies

$$\vec{\psi}=[L]\vec{I} \tag{4}$$

where $\vec{\psi}$ and $\vec{I}$ are $N_c\times 1$ column vectors. The jth element of $\vec{I}$ is the z-directed conduction current on the jth conductor. The ith element of $\vec{\psi}$ is the x-directed magnetic flux passing between a unit length of the ith conductor and the lower ground plane. If

$$\vec{I}=\frac{1}{\mu_0\epsilon_0}[C']\vec{\psi} \tag{5}$$

and if $[C']^{-1}$ exists, then (4) will imply that the desired relationship (2) is true. In the remainder of this section, we establish (5).

We formulate the magnetostatic problem with magnetic media in terms of the total electric current $J_T\boldsymbol{u}_z$ on the surfaces of the conducting cylinders and on the boundaries between different magnetic media. The total electric current on the conducting surfaces is the conduction current plus the magnetization current. The total electric current on the magnetic media boundaries is the magnetization current. The magnetic flux density $\boldsymbol{B}$ is given by

$$\boldsymbol{B}=\nabla\times\boldsymbol{A}=\nabla A_z\times\boldsymbol{u}_z \tag{6}$$

where A_z is the only component of $\boldsymbol{A}$ due to steady current flowing in the z direction. For the two-dimensional problem, it is given by

$$A_z(\rho)=\frac{\mu_0}{2\pi}\sum_{j=1}^{M}\int_{l_j}J_T(\rho')\ln\left(\frac{|\rho-\hat{\rho}'|}{|\rho-\rho'|}\right)dl' \tag{7}$$

where l_j denotes the jth interface. The first N_c interfaces are the surfaces of the N_c conductors. If the upper ground plane is present, the (N_c+1)th interface is the surface of this plane. The last N_d' interfaces are the magnetic media boundaries. Thus,

$$M = M_1 + N_d' \tag{8}$$

where M_1 is N_c if the upper ground plane is absent. If the upper ground plane is present, then M_1 is N_c+1. The number N_d' of magnetic media boundaries could be greater than $N_d - 1$ because these boundaries are arbitrarily shaped, not necessarily parallel to one another or to the lower ground plane. In (7), ρ' is the position vector of dl', and $\hat{\rho}'$ is the position vector of the image of dl' about the lower ground plane.

Substituting (7) into (6) and assuming that ρ is not on any of the interfaces $\{l_i\}$, we obtain

$$\boldsymbol{B}(\rho) = -\frac{\mu_0}{2\pi}\sum_{j=1}^{M}\int_{l_j} J_T(\rho')\left(\frac{\rho-\rho'}{|\rho-\rho'|^2} - \frac{\rho-\hat{\rho}'}{|\rho-\hat{\rho}'|^2}\right)\times \boldsymbol{u}_z\, dl'. \tag{9}$$

The limits of (9) as ρ approaches l_i from either side are

$$\boldsymbol{B}^{\pm}(\rho) = -\frac{\mu_0}{2\pi}\sum_{j=1}^{M} ⨍_{l_j} J_T(\rho')\left(\frac{\rho-\rho'}{|\rho-\rho'|^2} - \frac{\rho-\hat{\rho}'}{|\rho-\hat{\rho}'|^2}\right)\times \boldsymbol{u}_z\, dl' \mp \frac{\mu_0 J_T(\rho)}{2}(\boldsymbol{n}\times\boldsymbol{u}_z), \quad \begin{cases}\rho \text{ on } l_i\\ i=1,2,\cdots,M\end{cases} \tag{10}$$

where $⨍_{l_j}$ denotes the principal value of the integral over l_j, and $\boldsymbol{n}$ is a unit vector normal to l_i at ρ. Moreover, $\boldsymbol{B}^+(\rho)$ is $\boldsymbol{B}(\rho)$ on the side of l_i toward which $\boldsymbol{n}$ points, and $\boldsymbol{B}^-(\rho)$ is $\boldsymbol{B}(\rho)$ on the other side of l_i.

On the surface of the ith conductor, A_z is constant and is the x-directed magnetic flux ψ_i passing between a unit length of the ith conductor and the lower ground plane. Hence, similar to [1, eq. (A2)],

$$A_z(\rho) = \psi_i, \qquad \begin{cases}\rho \text{ on } l_i\\ i=1,2,\cdots,M_1.\end{cases} \tag{11}$$

If the upper ground plane is present, it is the (N_c+1)th conductor and, because A_z vanishes on it, $\psi_{N_c+1}=0$. Substitution of (7) into (11) gives

$$\frac{\mu_0}{2\pi}\sum_{j=1}^{M}\int_{l_j} J_T(\rho')\ln\left(\frac{|\rho-\hat{\rho}'|}{|\rho-\rho'|}\right)dl' = \psi_i, \qquad \begin{cases}\rho \text{ on } l_i\\ i=1,2,\cdots,M_1.\end{cases} \tag{12}$$

Continuity of the tangential component of magnetic intensity on the magnetic media boundaries requires that

$$\boldsymbol{n}\times\left[\frac{\boldsymbol{B}^+(\rho)}{\mu_i^+} - \frac{\boldsymbol{B}^-(\rho)}{\mu_i^-}\right] = 0, \qquad \begin{cases}\rho \text{ on } l_i\\ i=M_1+1, M_1+2,\cdots,M\end{cases} \tag{13}$$

where $\boldsymbol{n}$, $\boldsymbol{B}^+(\rho)$, and $\boldsymbol{B}^-(\rho)$ are the same as in (10). Furthermore, μ_i^+ is the permeability on the side of l_i toward which $\boldsymbol{n}$ points and μ_i^- is the permeability on the other side of l_i. Substituting (10) into (13) and then dividing (13) by $\boldsymbol{u}_z[1/\mu_i^+ - 1/\mu_i^-]$, we obtain

$$\frac{\mu_0}{2}\left[\frac{\dfrac{1}{\mu_i^+}+\dfrac{1}{\mu_i^-}}{\dfrac{1}{\mu_i^+}-\dfrac{1}{\mu_i^-}}\right]J_T(\rho) + \frac{\mu_0}{2\pi}\sum_{j=1}^{M} ⨍_{l_j} J_T(\rho') \cdot\left(\frac{\rho-\rho'}{|\rho-\rho'|^2} - \frac{\rho-\hat{\rho}'}{|\rho-\hat{\rho}'|^2}\right)\cdot\boldsymbol{n}\,dl' = 0, \quad \begin{cases}\rho \text{ on } l_i\\ i=M_1+1, M_1+2,\cdots,M.\end{cases} \tag{14}$$

If the ith conductor is of finite cross section, then l_i is a closed curve on which

$$\boldsymbol{u}_z J_T(\rho) = \frac{1}{\mu_0}\boldsymbol{n}\times\boldsymbol{B}^+(\rho) \tag{15}$$

$$\boldsymbol{u}_z J_c(\rho) = \frac{1}{\mu_i^+(\rho)}\boldsymbol{n}\times\boldsymbol{B}^+(\rho) \tag{16}$$

where $\boldsymbol{n}$ is the unit normal vector that points outward from the surface of the conductor. Furthermore, $\boldsymbol{B}^+(\rho)$ and μ_i^+ are, respectively, the magnetic flux density and the permeability just outside the conductor. In (16), $J_C(\rho)$ is the conduction current on the conductor. Equations (15) and (16) imply that

$$J_C(\rho) = \frac{\mu_0}{\mu_i^+(\rho)}J_T(\rho) \tag{17}$$

on the surface of the ith conductor, provided this conductor is of finite cross section.

If the ith conductor is an infinitesimally thin strip, then l_i runs from one edge of the strip to the other on which

$$J_C(\rho)\boldsymbol{u}_z = \boldsymbol{n}\times\left[\frac{\boldsymbol{B}^+(\rho)}{\mu_i^+(\rho)} - \frac{\boldsymbol{B}^-(\rho)}{\mu_i^-(\rho)}\right]. \tag{18}$$

Substitution of (10) for $\boldsymbol{B}^{\pm}(\rho)$ in (18) leads to

$$J_C(\rho) = \frac{\mu_0}{2}\left[\frac{1}{\mu_i^+(\rho)} + \frac{1}{\mu_i^-(\rho)}\right]J_T(\rho) + \frac{\mu_0}{2\pi}\left[\frac{1}{\mu_i^+(\rho)} - \frac{1}{\mu_i^-(\rho)}\right]\sum_{j=1}^{M} ⨍_{l_j} J_T(\rho') \cdot\left(\frac{\rho-\rho'}{|\rho-\rho'|^2} - \frac{\rho-\hat{\rho}'}{|\rho-\hat{\rho}'|^2}\right)\cdot\boldsymbol{n}\,dl' \tag{19}$$

on the surface of the ith conductor, provided this conductor is of zero thickness.

Now, consider the auxiliary electrostatic problem which has the same geometry as that of the present magnetostatic problem, but with relative permittivities ϵ_i'/ϵ_0 set equal to μ_0/μ_i. The formulation presented in [1] is, in fact, valid for dielectric media of arbitrary shape. The unit vector $\boldsymbol{u}_y$ in (11) of [1] should be replaced by $\boldsymbol{n}$ when the dielectric media are arbitrarily shaped. It is clear that (12), (14), (17), and (19) have the same mathematical forms as (9), (11), (15), and (17) of [1], respectively. Therefore, the solution of the magnetostatic problem can be related to that of the auxiliary electrostatic problem by

$$J_C^{(i)}(\rho') = \frac{1}{\mu_0\epsilon_0}\sigma_F^{(i)}(\rho'), \qquad i=1,2,\cdots,N_c \tag{20}$$

where $J_C^{(i)}(\rho')$ is the conduction current of the magnetostatic problem when $\psi_i=1$ is the only nonzero magnetic flux and $\sigma_F^{(i)}(\rho')$ is the free charge of the auxiliary electrostatic problem when the potential of the ith conductor is unity and all other conductors are grounded. Multiplying (20) by ψ_i and summing over i, we obtain

$$\sum_{i=1}^{N_c} J_C^{(i)}(\rho')\psi_i = \frac{1}{\mu_0\epsilon_0}\sum_{i=1}^{N_c}\sigma_F^{(i)}(\rho')\psi_i. \tag{21}$$

After noting that the right-hand side of (21) is $J_C(\rho')$, we integrate (21) over l_j to obtain (5) with the jith element of $[C']$ given by

$$C_{ji}' = \int_{l_j}\sigma_F^{(i)}(\rho')\,dl', \qquad i,j=1,2,\cdots,N_c. \tag{22}$$

Premultiplication of (5) by $[C']^{-1}$ yields

$$\vec{\psi} = \mu_0 \epsilon_0 [C']^{-1} \vec{I}. \tag{23}$$

The inverse of $[C']$ exists because $[C']$ is positive definite, which can be concluded from the fact that the electrostatic energy stored in the system is always greater than zero with nontrivial free charge distribution on the conductors. Comparison of (23) with (4) gives the desired relationship (2).

III. Conclusion

A simple relationship between the inductance matrix and the auxiliary capacitance matrix has been given. Thanks to this relationship, the computer code given in [1] and [3] for obtaining the capacitance matrix of the electrostatic problem can be used to obtain the inductance matrix of the magnetostatic problem.

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